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"String" formulation of the dynamics of the forward interest rate curve

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Abstract. We propose a formulation of the term structure of interest rates in which the forward curve is seen as the deformation of a string. We derive the general condition that the partial differential equations governing the motion of such string must obey in order to account for the condition of absence of arbitrage opportunities. This condition takes a form similar to a fluctuation-dissipation theorem, albeit on the same quantity (the forward rate), linking the bias to the covariance of variation fluctuations. We provide the general structure of the models that obey this constraint in the framework of stochastic partial (possibly non-linear) differential equations. We derive the general solution for the pricing and hedging of interest rate derivatives within this framework, albeit for the linear case (we also provide in the appendix a simple and intuitive derivation of the standard European option problem). We also show how the "string" formulation simplifies into a standard N-factor model under a Galerkin approximation.

PACS. 02.50.-r Probability theory, stochastic processes, and statistics – 05.40.+j Fluctuation phenomena, random processes, and Brownian motion – 89.90.+n Other areas of general interest to physicists

1 Introduction

Imagine that Julie wants to invest \$1 for two years [1]. She can devise two possible strategies. The first one is to put the money in a one-year bond at an interest rate r_1. At the end of the year, she must take her money and find another one-year bond, with interest rate r_2^1 which is the interest rate in one year on a loan maturing in two years. The final payoff of this strategy is simply $(1+r_1)(1+r_2^1)$. The problem is that Julie cannot know for sure what will be the one-period interest rate r_2^1 of next year. Thus, she can only estimate a return by guessing the expectation of r_2^1.

Instead of making two separate investments of one year each, Julie could invest her money today in a bond that pays off in two years with interest rate r_2. The final payoff is then $(1+r_2)^2$. This second strategy is riskless as she knows for sure her return. Now, this strategy can be reinterpreted along the line of the first strategy as follows. It consists in investing for one year at the rate r_1 and for the second year at a *forward rate* f_2. The forward rate is like the r_2^1 rate, with the essential difference that it is guaranteed: by buying the two-year bond, Julie can "lock in" an interest rate f_2 for the second year.

This simple example illustrates that the set of all possible bonds traded on the market is equivalent to the so-called forward rate curve. The forward rate $f(t,x)$ is thus the interest rate that can be contracted at time t for instantaneously riskless borrowing[1] or lending at time $t+x$. It is thus a function or curve of the time-to-maturity[2] x, where x plays the role of a "length" variable, that deforms with time t. Its knowledge is completely equivalent to the set of bond prices $P(t,x)$ at time t that expire at time $t+x$ (see Eq. (4) below). The shape of the forward rate curve $f(t,x)$ incessantly fluctuates as a function of time t. These fluctuations are due to a combination of factors, including future expectation of the short-term interest rates, liquidity preferences, market segmentation and trading. It is obvious that the forward rate $f(t,x+\delta x)$ for δx small can not be very different from $f(t,x)$. It is thus tempting to see $f(t,x)$ as a "string" characterized by a kind of tension which prevents too large local deformations that would not be financially acceptable. This superficial analogy is

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[1] "Instantaneous riskless" describes the fact that the forward rate is the rate that applies for a small time increment δt as seen from equation (4) below and is fixed during this time, thus being locally riskless.

[2] The maturity of a financial product is simply its lifetime. In other words, it is the time interval between the present and the time of extinction of the rights attached to the financial product.

in the follow up of the repetitious intersections between finance and physics, starting with Bachelier [2] who solved the diffusion equation of Brownian motion as a model of stock market price fluctuations five years before Einstein, continuing with the discovery of the relevance of Lévy laws for cotton price fluctuations by Mandelbrot [3] that can be compared with the present interest of such power laws for the description of physical and natural phenomena [4]. We could go on and cite many other examples. We investigate how to formalize mathematically this analogy between the forward rate curve and a string. In this goal, we formulate the term structure of interest rates as the solution of a stochastic partial differential equation (SPDE) [5], following the physical analogy of a continuous curve (string) whose shape moves stochastically through time.

The equation of motion of macroscopic physical strings is derived from conservation laws. The fundamental equations of motion of microscopic strings formulated to describe the fundamental particles [6] derive from global symmetry principles and dualities between long-range and short-range descriptions. Are there similar principles that can guide the determination of the equations of motion of the more down-to-earth financial forward rate "strings"?

The situation is a priori much more difficult than in physics as illustrated by the following picturial analogy quoted from the journalist N. Dunbar at Financial Products magazine. Suppose that in the middle ages, before Copernicus and Galileo, the Earth really was stationary at the centre of the universe, and only began moving later on. Imagine that during the nineteenth century, when everyone believed classical physics to be true, that it really was true, and quantum phenomena were non-existent. These are not philosophical musings, but an attempt to portray how physics might look if it actually behaved like the financial markets. Indeed, the financial world is such that any insight is almost immediately used to trade for a profit. As the insight spreads among traders, the "universe" changes accordingly. As Soros has pointed out, market players are "actors observing their own deeds". As Derman, head of quantitative strategies at Goldman Sachs, puts it, in physics you are playing against God, who does not change his mind very often. In finance, you are playing against God's creatures, whose feelings are ephemeral, at best unstable, and the news on which they are based keep streaming in. Value clearly derives from human beings, while mass, charge and electromagnetism apparently do not. This has led to suggestions that a fruitful framework to study finance and economy is to use evolutionary models inspired from biology and genetics.

This does not however guide us much for the determination of "fundamental" equations, if any. Here, we propose to use the condition of absence of arbitrage opportunity[3] and show that this leads to strong constraints on the structure of the governing equations. The basic idea is that, if there are arbitrage opportunities (free lunches), they cannot live long or must be quite subtle, otherwise traders would act on them and arbitrage them away. The no-arbitrage condition is an idealization of a self-consistent dynamical state of the market resulting from the incessant actions of the traders (arbitragers). It is not the out-of-fashion equilibrium approximation sometimes described but rather embodies a very subtle cooperative organization of the market.

We consider this condition as the fundamental backbone for the theory. The idea to impose this requirement is not new and is in fact the prerequisite of most models developed in the academic finance community. However, applying it in the present context is new. Modigliani and Miller [7,8] have indeed emphasized the critical role played by arbitrage in determining the value of securities. It is sometimes suggested that transaction costs and other market imperfections make irrelevant the no-arbitrage condition [9]. Let us address briefly this question before presenting our results.

Transaction costs in option replication and other hedging activities[4] have been extensively investigated since they (or other market "imperfections") clearly disturb the risk-neutral argument and set option theory back a few decades. Transaction costs induce, for obvious reasons, dynamic incompleteness, thus preventing valuation as we know it since Black and Scholes [10]. However, the most efficient dynamic hedgers (market makers) incur very small transaction costs when owning options[5] (see the appendix for a definition of options). These specialized market makers compete with each other to provide liquidity in option instruments, and maintain inventories in them. They rationally limit their dynamic replication to their residual exposure, not their global exposure. In addition, the fact that they do not hold options until maturity greatly reduces their costs of dynamic hedging. They have an incentive in the acceleration of financial intermediation. Furthermore, as options are rarely replicated until maturity, the expected transaction costs of the short options depend mostly on the dynamics of the order flow in the option markets -not on the direct costs of transacting. The conclusion is that transaction costs are a fraction of what has been assumed to be in the literature [11]. For the efficient operators (and those operators only), markets are more dynamically complete than anticipated. This is not true for a second category of traders, those who merely purchase or sell financial instruments that are subjected to dynamic hedging. They, accordingly, neither are equipped for dynamic hedging, nor have the need for it, thanks to the existence of specialized and more efficient market makers. The examination of their transaction costs in the event of their decision to dynamically replicate their options is of no true theoretical contribution.

A second important point is that the existence of transaction costs should not be invoked as an excuse for

[3] Arbitrage, also known as the Law of One Price, states that two assets with identical attributes should sell for the same price and so should the same asset trading in two different markets. If the prices differ, a profitable opportunity arises to sell the asset where it is overpriced and to buy it where it is underpriced.

[4] Finance is all about risks but some risks can be hedged, *i.e.* offset, by trading in different financial instruments.

[5] In a nutshell, an option is an insurance for buying or selling.

disregarding the no-arbitrage condition but rather should be constructively invoked to study its impacts on the models. We should however caution that a still open and unresolved fundamental question is the nature of the "perturbation" when going from the perfect no-arbitrage condition to weak-arbitrage in presence of market imperfections. We implicitely assume in this work that the perturbation is smooth, i.e. the perturbed real dynamics is close to the perfect case. We cannot however exclude a singular perturbation.

This work expands our previous work [12] which introduced stochastic strings multiplied by volatility functions to shock forward rates. This theory models the true dynamics and not only the risk-neutral dynamics. The risk component is taken into account by the introduction of a pricing kernel. In the present formulation, we restrict the description to the risk-neutral dynamics by putting the pricing kernel to zero. Re-introducing the pricing kernel of risks does not pose any difficulty. Our present approach directly deals with the SPDE equation for the forward rates in contrast to the shocks that drive the forward rate curve that were treated in reference [12]. Our model provides a further extension of the term structure model of Heath, Jarrow and Morton [14] and is as parsimonious and tractable as the traditional HJM model, but is capable of generating a much richer class of dynamics and shapes of the forward rate curve. Its main motivation is to address the interplay between external factors represented by stochastic components (noise) and possible non-linear dynamics.

2 Definitions

We postulate the existence of a stochastic discount factor (SDF) that prices all assets in this economy and denote it by M. This process is also termed the pricing kernel, the pricing operator, or the state price density. We use these terms interchangeably. Reference [15] is an excellent reference for the theory behind the SDF. It is well known that assuming that no dynamic arbitrage trading strategies can be implemented by trading in the financial securities issued in the economy is roughly equivalent to the existence of a strictly positive SDF. For no arbitrage opportunities to exist, the product of M with the value process of any investment strategy must be a martingale[6]. Under an adequate definition of the space of admissible trading strategies, the product MV is a martingale, where V is the value process of any admissible self-financing trading strategy implemented by trading on financial securities. Then,

$$V(t) = E_t \left[V(s) \frac{M(s)}{M(t)} \right], \tag{1}$$

where s is a future date and $E_t[x]$ denotes the mathematical expectation of x taken at time t. In particular, we require that a bank account and zero-coupon discount bonds of all maturities satisfy this condition.

A security is referred to as a (floating-rate) bank account, if it is "locally riskless"[7]. Thus, the value at time t, of an initial investment of $B(0)$ units in the bank account that is continuously reinvested, is given by the following process

$$B(t) = B(0) \exp \left\{ \int_0^t r(s) ds \right\}, \tag{2}$$

where $r(t)$ is the instantaneous nominal interest rate.

We further assume that, at any time t, riskless discount bonds of all *maturity dates* s trade in this economy and let $P(t,s)$ denote the time t price of the s maturity bond. We require that $P(s,s) = 1$, that $P(t,s) > 0$ and that $\partial P(t,s)/\partial s$ exists.

Instantaneous forward rates at time t for all *times-to-maturity* $x > 0$, $f(t,x)$, are defined by

$$f(t,x) = -\frac{\partial \log P(t, t+x)}{\partial x}, \tag{3}$$

which is the rate that can be contracted at time t for instantaneously riskless borrowing or lending at time $t+x$. We require that the initial forward curve $f(0,x)$, for all x, be continuous.

Equivalently, from the knowledge of the instantaneous forward rates for all times-to-maturity between 0 and time $s-t$, the price at time t of a bond with maturity s can be obtained by

$$P(t,s) = \exp \left\{ - \int_0^{s-t} f(t,x) dx \right\}. \tag{4}$$

Forward rates thus fully represent the information in the prices of all zero-coupon bonds.

The spot interest rate at time t, $r(t)$, is the instantaneous forward rate at time t with time-to-maturity 0,

$$r(t) = f(t,0). \tag{5}$$

For convenience, we model the dynamics of forward rates. Clearly, we could as well model the dynamics of bond prices directly, or even the dynamics of the yields to maturity of the zero-coupon bonds. We use forward rates with fixed *time-to-maturity* rather than fixed *maturity date*. The model of HJM starts from processes for forward rates with a fixed maturity date. This is different from what we do. If we use a "hat" to denote the forward rates modeled by HJM,

$$\hat{f}(t,s) = f(t, s-t) \tag{6}$$

or, equivalently,

$$f(t,x) = \hat{f}(t, t+x) \tag{7}$$

[6] Technically, recall that a martingale is a family of random variables $\xi(t)$ such that the mathematical expectation of the increment $\xi(t_2) - \xi(t_1)$ (for arbitrary $t_1 < t_2$), conditioned on the past values $\xi(s)$ ($s \le t_1$), is zero. The drift of a martingale is thus zero. This is different from the Markov process, which is better known in the physical community, defined by the independence of the next increment on past values.

[7] A security is "locally riskless" if, over an instantaneous time interval, its value varies deterministically. It may still be random, but there is no Brownian term in its dynamics.

for fixed s. Brace and Musiela [16] define forward rates in the same fashion. Miltersen, Sandmann and Sondermann [17], and Brace, Gatarek and Musiela [18] use definitions of forward rates similar to ours, albeit for non-instantaneous forward rates. Modelling forward rates with fixed time-to-maturity is more natural for thinking of the dynamics of the entire forward curve as the shape of a string evolving in time. In contrast, in HJM, forward rate processes disappear as time reaches their maturities. Note, however, that we still impose the martingale condition on bonds with fixed *maturity date*, since these are the financial instruments that are actually traded.

3 Stochastic strings as solutions of SPDE's

In a nutshell, the contribution of this paper consists in modelling the dynamical evolution of the forward rate curve by stochastic *partial* differential equations (SPDE's) [5]. In the context of continuous-time finance, this is the most natural and general extension that can be performed[8].

Financial and economic time series are often described to a first degree of approximation as random walks, following the precursory work of Bachelier [2] and Samuelson [19]. A random walk is the mathematical translation of the trajectory followed by a particle subjected to random velocity variations. The analogous physical system described by SPDE's is a *stochastic string*. The length along the string is the time-to-maturity and the string configuration (its transverse deformation) gives the value of the forward rate $f(t,x)$ at a given time for each time-to-maturity x. The set of admissible dynamics of the configuration of the string as a function of time depends on the structure of the SPDE. Let us for the time being restrict our attention to SPDE's in which the highest derivative is second order. This second order derivative has a simple physical interpretation: the string is subjected to a tension, like a piano chord, that tends to bring it back to zero transverse deformation. This tension forces the "coupling" among different times-to-maturity so that the forward rate curve is at least continuous. In principle, the most general formulation would consider SPDE's with terms of arbitrary derivative orders[9]. However, it is easy to show that the tension term is the dominating restoring force, when present, for deformations of the string (forward rate curve) at long "wavelengths", *i.e.* for slow variations along the time-to-maturity axis. Second order SPDE's are thus generic in the sense of a systematic expansion[10].

In the framework of second order SPDE's, we consider *hyperbolic*, *parabolic* and *elliptic* SPDE's, to characterize the dynamics of the string along two directions : inertia or mass, and viscosity or subjection to drag forces. A string that has "inertia" or, equivalently, "mass" per unit length, along with the tension that keeps it continuous, is characterized by the class of *hyperbolic* SPDE's. For these SPDE's, the highest order derivative in time has the same order as the highest order derivative in distance along the string (time-to-maturity). As a consequence, hyperbolic SPDE's present wave-like solutions, that can propagate as pulses with a "velocity". In this class, we find the so-called "Brownian sheet" which is the direct generalization of Brownian motion to higher dimensions, that preserves continuity in time-to-maturity. The Brownian sheet is the surface spanned by the string configurations as time goes on. The Brownian sheet is however non-homogeneous in time-to-maturity, which led us to examine other processes.

If the string has no inertia[11], its dynamics are characterized by *parabolic* SPDE's. These stochastic processes lead to smoother diffusion of shocks through time, along time-to-maturity.

Finally, we mention the third class of SPDE's of second-order, namely elliptic partial differential equations. Elliptic SPDE's give processes that are differentiable both in x and t. Therefore, in the strict limit of continuous trading, these stochastic processes correspond to locally riskless interest rates.

For the sake of completeness and clarity, we briefly summarize useful facts about PDE's (See for instance Ref. [20] and, in particular, their classification and the intuitive meaning behind it). We restrict our discussion to two-dimensional examples. Their general form reads

$$\begin{aligned} A(t,x)\ \frac{\partial^2 f(t,x)}{\partial t^2} &+ 2B(t,x)\frac{\partial^2 f(t,x)}{\partial t \partial x} \\ &+C(t,x)\frac{\partial^2 f(t,x)}{\partial x^2} = \\ &F(t,x,f(t,x),\frac{\partial f(t,x)}{\partial t},\frac{\partial f(t,x)}{\partial x},S), \end{aligned} \tag{8}$$

where $f(t,x)$ is the forward rate curve. $S(t,x)$ is the "source" term that will be generally taken to be Gaussian white noise $\eta(t,x)$ characterized by the covariance

$$\mathrm{Cov}\left[\eta(t,x),\ \eta(t',x')\right] = \delta(t-t')\ \delta(x-x'), \tag{9}$$

where δ denotes the Dirac distribution. Expression (8) is the most general second-order SPDE in two variables. For arbitrary non-linear terms in F, the existence of solutions

[8] Further extensions will include fractional differential equations and integro-differential equations, including jump processes.

[9] Higher order derivatives also have an intuitive physical interpretation. For instance, going up to fourth order derivatives in the SPDE correspond to the dynamics of a *beam*, which has bending elastic modulus tending to restore the beam back to zero deformation, even in absence of tension.

[10] There are situations where the tension can be made to vanish (for instance in the presence of a rotational symmetry) and then the leading term in the SPDE becomes the fourth order "beam" term.

[11] Or if the inertia term is negligible compared to the drag term proportional to the first time derivative (so-called overdamped dynamics).

is not warranted and a case by case study must be performed. For the cases where F is linear, the solution $f(t,x)$ exists and its uniqueness is warranted once "boundary" conditions are given, such as, for instance, the initial value of the function $f(0,x)$ as well as any constraints on the particular form of equation (8).

Equation (8) is defined by its *characteristics*, which are curves in the (t,x) plane that come in two families of equation:

$$Adt = (B + \sqrt{B^2 - AC})dx, \tag{10}$$

$$Adt = (B - \sqrt{B^2 - AC})dx. \tag{11}$$

These characteristics are the geometrical loci of the propagation of the boundary conditions.

Three cases must be considered.

• When $B^2 > AC$, the characteristics are real curves and the corresponding SPDE's are called "hyperbolic". For such hyperbolic SPDE's, the natural coordinate system is formed from the two families of characteristics. Expressing (8) in terms of these two natural coordinates λ and μ, we get the "normal form" of hyperbolic SPDE's:

$$\frac{\partial^2 f}{\partial\lambda\partial\mu} = P(\lambda,\mu)\frac{\partial f}{\partial\lambda} + Q(\lambda,\mu)\frac{\partial f}{\partial\mu} + R(\lambda,\mu)f + S(\lambda,\mu). \tag{12}$$

The special case $P = Q = R = 0$ with $S(\lambda,\mu) = \eta(\lambda,\mu)$ corresponds to the so-called Brownian sheet, well studied in the mathematical literature as the 2D *continuous* generalization of the Brownian motion.

• When $B^2 = AC$, there is only one family of characteristics, of equation

$$Adt = Bdx. \tag{13}$$

Expressing (8) in terms of the natural characteristic coordinate λ and keeping x, we get the "normal form" of parabolic SPDE's:

$$\frac{\partial^2 f}{\partial x^2} = K(\lambda,\mu)\frac{\partial f}{\partial\lambda} + L(\lambda,\mu)\frac{\partial f}{\partial x} + M(\lambda,\mu)f + S(\lambda,\mu). \tag{14}$$

The diffusion equation, well-known to be associated to the Black-Scholes option pricing model, is of this type. The main difference with the hyperbolic equations is that it is no more invariant with respect to time-reversal $t \to -t$. Intuitively, this is due to the fact that the diffusion equation is not conservative, the information content (negentropy) continually decreases as time goes on.

• When $B^2 < AC$, the characteristics are *not* real curves and the corresponding SPDE's are called "elliptic". The equations for the characteristics are complex conjugates of each other and we can get the "normal form" of elliptic SPDE's by using the real and imaginary parts of these complex coordinates $z = u \pm iv$:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = T\frac{\partial f}{\partial u} + U\frac{\partial f}{\partial v} + Vf + S. \tag{15}$$

There is a deep connection between the solution of elliptic SPDE's and analytic functions of complex variables.

We have shown [12] that hyperbolic and parabolic SPDE's provide processes reducing locally to standard Brownian motion at fixed time-to-maturity, while elliptic SPDE's give locally riskless time evolutions. Basically, this stems from the fact that the "normal forms" of second-order hyperbolic and parabolic SPDE's involve a first-order derivative in time, thus ensuring that the stochastic processes are locally Brownian in time. In contrast, the "normal form" of second-order elliptic SPDE's involve a second-order derivative with respect to time, which is the cause for the differentiability of the process with respect to time. Any higher order SPDE will be Brownian-like in time if it remains of order one in its time derivatives (and higher-order in the derivatives with respect to x).

4 No-arbitrage condition: derivation of the general condition

We now proceed to derive the general condition that the forward rate equation must obey to be compatible with the no-arbitrage constraint.

From (4), we get

$$d_t \log P(t,s) = f(t,x)\ dt - \int_0^x dy\ d_t f(t,y), \tag{16}$$

where $x \equiv s - t$. We need the expression of $\frac{dP(t,s)}{P(t,s)}$ which is obtained from (16) using Ito's calculus [13]. In order to get Ito's term in the drift, recall that it results from the fact that, if f is stochastic, then

$$d_t F(f) = \frac{\partial F}{df} d_t f + \frac{1}{2}\int dx \int dx' \frac{\partial^2 F}{\partial f(t,x)\partial f(t,x')} \times \mathrm{Cov}\left[d_t f(t,x), d_t f(t,x')\right], \tag{17}$$

where $\mathrm{Cov}\left[d_t f(t,x), d_t f(t,x')\right]$ is the covariance of the time increments of $f(t,x)$.

Using this Ito's calculus, we obtain

$$\frac{dP(t,s)}{P(t,s)} = \left[dt\ f(t,x) - \int_0^x dy\ E_{t,d_t f}(t,y) + \frac{1}{2}\int_0^x dy \int_0^x dy'\ \mathrm{Cov}\left[d_t f(t,y) d_t f(t,y')\right]\right] - \int_0^x dy\ [d_t f(t,y) - E_{t,d_t f}(t,y)]. \tag{18}$$

We have explicitely taken into account the fact that $d_t f(t,x)$ may have in general a non-zero drift, *i.e.* its expectation

$$E_{t,d_t f}(t,x) \equiv E_t\left[d_t f(t,x) | f(t,x)\right] \tag{19}$$

conditioned on $f(t,x)$ is non-zero.

The no-arbitrage condition for buying and holding bonds implies that PM is a martingale in time, for any bond price P. Technically this amounts to imposing that the drift of PM be zero:

$$f(t,x) = f(t,0) + \int_0^x dy\ \frac{E_{t,d_t f}(t,y)}{dt} - \frac{1}{2}\int_0^x dy \int_0^x dy'\ c(t,y,y') + o(1), \quad (20)$$

assuming that $d_t f(t,x)$ is not correlated with the stochastic process driving the pricing kernel and using the definitions

$$c(t,y,y')dt = \mathrm{Cov}[d_t f(t,y) d_t f(t,y')], \quad (21)$$

and $r(t) = f(t,0)$ as given by (5). In (20), the notation $o(1)$ designs terms of order dt taken to a positive power.

Expression (20) is the fundamental constraint that a SPDE for $f(t,x)$ must satisfy in order to obey the no-arbitrage requirement. As in other formulations, this condition relates the drift to the volatility.

It is useful to parameterize, without loss of generality,

$$\frac{E_{t,d_t f}(t,x)}{dt} = \frac{\partial f(t,x)}{\partial x} + h(t,x), \quad (22)$$

where $h(t,x)$ is *a priori* arbitrary. The usefulness of this parameterization (22) stems from the fact that it allows us to get rid of the terms $f(t,x)$ and $f(t,0)$ in (20). Indeed, they cancel out with the integral over y of $\frac{E_{t,d_t f}(t,y)}{dt}$. Taking the derivative with respect to x of the no-arbitrage condition (20), we obtain

$$h(t,x) = -\frac{1}{2}\int_0^x\ [c(t,y,x) + c(t,x,y)]. \quad (23)$$

In sum, we have the following constraint that the SPDE for $f(t,x)$ must satisfy

$$E_t\left[d_t f(t,x)|f(t,x)\right] = dt\ \frac{\partial f(t,x)}{\partial x} - \frac{1}{2}\int_0^x dy \times \Bigg(\mathrm{Cov}[d_t f(t,y) d_t f(t,x)] + \mathrm{Cov}[d_t f(t,x) d_t f(t,y)]\Bigg), (24)$$

which, for symmetric covariance, lead to

$$E_t\ \ [d_t f(t,x)|f(t,x)] = dt\ \frac{\partial f(t,x)}{\partial x} - \int_0^x dy\ \mathrm{Cov}[d_t f(t,y) d_t f(t,x)]. \quad (25)$$

This expression (25) is reminiscent of the fluctuation-dissipation theorem in the Langevin formulation of the Brownian motion [21], linking the drag coefficient (analog to the l.h.s.) to the integral over time of the correlation function of the fluctuation forces (analog of the second term[12] of the r.h.s.). In the usual fluctuation-dissipation theorem, the drag coefficient is determined self-consistently as a function of the amplitude and correlation of the fluctuating force in order to be compatible with the equilibrium distribution. Similarly, here the no-arbitrage condition determines self-consistently the conditional drift from the covariance of the fluctuations. In contrast, notice however that the same quantity f enters in both sides of (24) and (25). Thus, the no-arbitrage condition imposes a condition on the structure of the partial differential equations that govern the dynamical evolution of the forward rates $f(t,x)$.

[12] The first term of the r.h.s. is a drift term that disappear by a galilean transformation of frame.

5 General structure of the SPDE's compatible with the no-arbitrage condition

To derive the structure of the SPDE's compatible with the no-arbitrage condition (24), we come back to the formulation of [12] and parameterize the time increment of the forward rate as

$$d_t f(t,x) = \alpha(t,x)dt + \sigma(t,x) d_t Z(t,x), \quad (26)$$

where $Z(t,x)$ is a infinite dimensional stochastic process which is continuous in x and t and $\alpha(t,x)$ and $\sigma(t,x)$ are *a priori* arbitrary functions of $f(t,x)$.

Using this formulation (26), we have shown previously [12] that the condition of no arbitrage leads to the following dynamics for the forward rates

$$d_t f(t,x) = dt\Bigg(\frac{\partial f(t,x)}{\partial x} + A(t,x)\Bigg) + \sigma(t,x)\ d_t Z(t,x), \quad (27)$$

where

$$A(t,x) = \sigma(t,x)\Bigg(\int_0^x dy\ \sigma(t,y) c_Z(t,y,x)\Bigg), \quad (28)$$

and

$$c_Z(t,y,y')\ dt \equiv \mathrm{Cov}\left[d_t Z(t,y), d_t Z(t,y')\right]. \quad (29)$$

We stress that, by construction, the form (27) with (28) and (29) automatically satisfies the condition (24).

In order to get the form of the allowed SPDE's for $f(t,x)$, we recall the requirements that are convenient to impose Z, without loss of generality.

1. $Z(t,x)$ is continuous in x at all times t;
2. $Z(t,x)$ is continuous in t for all x;
3. $Z(t,x)$ is a martingale in time t, $E\left[d_t Z(t,x)\right] = 0$, for all x;
4. The variance of the increments, $\mathrm{Var}\left[d_t Z(t,x)\right]$, does not depend on t or x;
5. The correlation of the increments, $\mathrm{Corr}\left[d_t Z(t,x), d_t Z(t,x')\right]$, does not depend on t.

Then, a fairly general stochastic function $Z(t,x)$ obeying these requirements is such that [12]

$$d_t Z(t,x) = dt\ \frac{1}{\sqrt{j(x)}}\int_0^{j(x)} dy\ \eta(t,y), \quad (30)$$

where $\eta(t,x)$ is a Gaussian white noise characterized by the covariance (9). This leads to the following covariance function for $Z(t,x)$:

$$c_Z(t,y,y') = \sqrt{\frac{j(x)}{j(x')}} \quad \text{for} \quad j(x) < j(x'), \tag{31}$$

and the roles of x and x' in (31) are reversed if $j(x) > j(x')$.

Inserting (30) in (27), we get

$$\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x} = \sigma(t,x)\left(\int_0^x dy\ \sigma(t,y)\sqrt{\frac{j(y)}{j(x)}}\right) + \frac{\sigma(t,x)}{\sqrt{j(x)}}\int_0^{j(x)} dy\ \eta(t,y). \tag{32}$$

Multiplying both sides of this equation by $\frac{\sqrt{j(x)}}{\sigma(t,x)}$ and taking the partial derivative with respect to x finally yields

$$\frac{\partial}{\partial x}\left[\frac{\sqrt{j(x)}}{\sigma(t,x)}\left(\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\sqrt{j(x)}\int_0^x dy\ \sigma(t,y)\sqrt{\frac{j(y)}{j(x)}}\right] + \sqrt{|\frac{dj(x)}{dx}|}\ \eta(t,x), \tag{33}$$

where we have used that $\eta(t,j(x)) = \eta(t,x)/\sqrt{|dj(x)/dx|}$. This provides a first class of SPDE, which is in general non-linear since the volatility $\sigma(t,x)$ can be an arbitrary function of $f(t,x)$.

An even more general class of SPDE's for $f(t,x)$ is obtained by using the most general stochastic function $Z(t,x)$ obeying the requirements $1-5$ [12]:

$$d_t Z(t,x) = dt\ \frac{1}{\sqrt{l(x)}}\int_0^{j(x)} dy\ \sqrt{\frac{d}{dy} l\left([j]^{-1}\right)(y)}\ \eta(t,y). \tag{34}$$

The correlation of the increments is

$$c_Z(t,x,x') = \sqrt{\frac{l(x)}{l(x')}}, \quad \text{if} \quad j(x) < j(x'). \tag{35}$$

The role of x and x' in (35) are inverted if $j(x) > j(x')$. This provides a generalization to (30) since a different function appears in the correlation function and in the inequality condition on x and x'.

Inserting (34) in (27), we get

$$\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x} = \sigma(t,x) \times\left(\int_0^x dy\ \sigma(t,y)\sqrt{\frac{l(y)}{l(x)}}\right) + \frac{\sigma(t,x)}{\sqrt{l(x)}} \times \int_0^{j(x)} dy\ \sqrt{\frac{d}{dy} l\Big([j]^{-1}\Big)(y)\ \eta(t,y)}\Bigg). \tag{36}$$

Multiplying both sides of this equation by $\frac{\sqrt{l(x)}}{\sigma(t,x)}$ and taking the partial derivative with respect to x finally yields

$$\frac{\partial}{\partial x}\ \left[\frac{\sqrt{l(x)}}{\sigma(t,x)}\left(\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\sqrt{j(x)}\int_0^x dy\ \sigma(t,y)\sqrt{\frac{l(y)}{l(x)}}\right] + \sqrt{\frac{dl(x)}{dx}}\sqrt{|\frac{dj(x)}{dx}|}\ \eta(t,x). \tag{37}$$

This provides the most general class of SPDE describing the dynamical evolution of the forward rates $f(t,x)$ due to the interplay of a stochastic forcing η and non-linearities embodied in $\sigma(t,x)$.

According to the classification briefly summarized above, these equations (33) and (37) are both of the hyperbolic class. It is noteworthy that the no-arbitrage condition excludes the parabolic class. In contrast, physical strings obey hyperbolic equations for zero or small dissipation and parabolic equations in the over-damped limit. *A posteriori*, it is not surprising that we find that the no-arbitrage condition, which implies the absence of market friction, corresponds to the first class. Notice that the term $\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x}$ can be replaced by $\frac{\partial f(t,x)}{\partial t}$ when changing the variables (t,x) to $(t, s = t+x)$ and the l.h.s. of (37) become of the form $\frac{\partial}{\partial s}\left[\frac{\sqrt{l(s)}}{\sigma(t,s)}\frac{\partial \hat{f}(t,s)}{\partial t}\right]$.

These equations (33) and (37) are characterized by partial derivatives and two source terms in the r.h.s., the first one being locally adapted and the second one corresponding to the influence of external noise. These equations can be explicitely solved if σ is specified and is independent of $f(t,x)$, thus keeping the equations linear, as done in reference [12]. The problem of finding a solution of these equations when $\sigma(t,x)$ depends on $f(t,x)$ is much more complex and belongs to the vast class of generally unsolved non-linear partial differential equations. Such equations have been encountered in many different areas, in particular some apparently particularly simple nonlinear PDE's have been found to exhibit the most complex phenomenology one can imagine. A vivid example is the Navier-Stokes equation of fluid motion leading to fluid turbulence when its single parameter (viscosity) goes to zero [22]. Extrapolating on these superficial analogies, we conjecture that it might be possible to postulate a simple form for the nonlinear dependence of σ as a function of $f(t,x)$, such that one or two real parameters might embody the full phenomenology of observed forward rate statistics.

6 A parametric example

Empirical observation shows that correlation between two forward rates, with maturities separated by a given interval, increases with maturity. We have shown previously [12] that the parameterization $j(x) = e^{i(x)}$ leads to the simple and general condition that $i(x)$ must be a function which increases more slowly than x, *i.e.* $i(x)$ must be concave (downward). In order words, $j(x)$ must grow slower than an exponential. Any expression like $j(x) = \exp(\kappa x^\alpha)$ with an exponent $0 < \alpha < 1$ qualifies. This corresponds to

$$c_Z(t,x,y) = e^{-\kappa|x^\alpha - y^\alpha|}. \quad (38)$$

Reporting this choice in (33) yields the following SPDE:

$$\frac{\partial}{\partial x}\left[\frac{e^{(\kappa/2)x^\alpha}}{\sigma(t,x)}\left(\frac{\partial f(t,x)}{\partial t} - \frac{\partial f(t,x)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\int_0^x dy\ \sigma(t,y) e^{(\kappa/2)y^\alpha}\right] + \sqrt{\kappa\alpha}\, x^{\frac{\alpha-1}{2}} e^{(\kappa/2)x^\alpha}\ \eta(t,x). \quad (39)$$

7 Reduction to N-factors models

Let us now show how our model naturally encompasses the usual HJM formulation of forward interest rates in terms of N factors. The idea is that N factors models can be obtained as truncations at the order N in a way similar to a galerkin approximation, corresponding to a finite "resolution", of an infinite expansion over the eigenfunctions of the operator defining the partial differential equation. This point of view allows one to clearly see both the relationship with previous formulations and their limitations. It also gives a justification to N factor models in the following sense. As the stochastic string description constitutes the most general description of the forward rate curve, the fact that the N factors models emerge naturally by a truncation of this general formulation, amounting to limit the resolution of the fluctuations in the time of maturity axis, justifies their mathematical status as simply a degree of approximation of an ideal general description.

To keep the exposition as general as possible, let us call $\mathcal{L}$ the *linear* operator defined by

$$\mathcal{L}f(t,x) = \eta(t,x). \quad (40)$$

We simplify the problem by restricting our analysis to linear SPDE's. This allows us to use the general property of linear operators that the Green function $G(t,x|v,y)$ defined by

$$\mathcal{L}G(t,x|v,y) = \delta(t-v)\ \delta(x-y) \quad (41)$$

can be expressed as [20]

$$G(t,x|v,y) = \sum_{n=1}^{\infty} l_n(v,t)\ \psi_n(x)\ \phi_n(y). \quad (42)$$

The $\psi_n(x)$ are the eigenfunctions of $\mathcal{L}$. The $\phi_n(x)$ are the eigenfunctions of the adjoint operator $\mathcal{L}^*$, which is the same as $\mathcal{L}$ for the hyperbolic string wave equations considered above (self-adjoint case) (but would not be the same for a parabolic string equation). The $l_n(v,t)$ are a set of functions which depend on $\mathcal{L}$. Now, the eigenfunctions of $\mathcal{L}$ form an orthogonal basis on which one can expand the source term $\eta(t,x)$:

$$\eta(t,x) = \sum_{j=1}^{\infty} \eta_j(t)\ \psi_j(x). \quad (43)$$

The delta-covariance property of $\eta(t,x)$ allows us to choose that of the $\eta_j(t)$ as follows:

$$\mathrm{Cov}[\eta_j(t), \eta_k(t')] = g_{jk}(t)\ \delta(t-t'), \quad (44)$$

where $g_{jk}(t)$ depends on the form of the operator. Using the general solution of (40) which reads

$$f(t,x) = f(0,x) + \int_0^t dv \int_{-\infty}^{\infty} dy\ G(t,x|v,y)\ \eta(v,y), \quad (45)$$

we obtain

$$f(x,t) = \int_0^t dv \int_{-\infty}^{\infty} dy \sum_{n=1}^{\infty} l_n(v,t)\ \psi_n(x)\ \phi_n(y) \sum_{j=1}^{\infty} \eta_j(v)\ \psi_j(y). \quad (46)$$

By the orthogonality condition $\int_{-\infty}^{\infty} dy \phi_n(y)\ \psi_j(y) = \delta_{nj}$, we obtain

$$f(x,t) = \int_0^t dv \sum_{n=1}^{\infty} l_n(v,t)\ \psi_n(x) \eta_n(v). \quad (47)$$

We recognize a N-factor model by truncating this sum at $n = N$ and with the identification

$$\sqrt{g_{nn}(t)}\ l_n(v,t)\ \psi_n(x) \equiv \sigma_n(t,x), \quad (48)$$

and

$$\frac{\eta_n(t)dt}{\sqrt{g_{nn}(t)}} \equiv d_t W_n(t). \quad (49)$$

Since the functions $\psi_n(x)$ are more and more complicated or convoluted as the order n increases, the truncation at a finite order N indeed corresponds to a finite resolution in the driving of the forward rate curve. For instance, $\psi_1(x)$ can be a constant, $\psi_2(x)$ can be a parabola with a single maximum, $\psi_3(x)$ can be a quartic (two up maxima and one down maximum), etc.

8 Option pricing and replication

In this section, we derive the general equation for the pricing and hedging of interest rate derivatives. Our derivation is restricted to linear SPDE's, *i.e.* to the cases where $\sigma(t,x)$ is not an explicit function of the forward rates f.

8.1 The general case

In general, European term structure derivatives in our model have a payoff function of the form

$$C(t, \{P(s)\}, r(t)), \quad (50)$$

where the argument $\{P(s)\}$ indicates that the payoff depends on the price at time s of all bonds of different time of maturities, *i.e.* on the full forward rate curve[13]. The problem we address is that of hedging this claim by trading bonds. Hedging these claims in general implies trading in an infinity of bonds, so that the claim's price at time t will be a function of the entire forward rate curve.

There is thus in general no hope of being able to perfectly hedge interest rate contingent claims with a finite number of bonds. We therefore extend the space of admissible trading strategies to include density valued portfolios.

We denote by $h(t,s)$ the density (number) of bonds held in the portfolio at time t, with maturity s, and let $g(t)$ be the amount invested in the bank account. Then, the value of an investment strategy is

$$V(t) = g(t)B(t) + \int_t^{\infty} h(t,u)P(t,u)du.$$

Let us call V_1 the value of the portfolio made of the contingent claim, with value process C, financed at the risk-free rate $r(t)$, and V_2 that of the replicating portfolio consisting of bonds and of money invested in the bank account at the risk-free interest rate $r(t)$ (equal to the the spot rate $f(t,0)$ which may fluctuate but is risk-less in the sense that it gives the instantaneous payoff of cash invested in the bank account). At time $0 \leq t \leq T$, the variation of V_1 with time, discounted by the risk-free interest rate, is

$$d_t V_1(t) = d_t C(t) - r(t)C(t)dt. \quad (51)$$

The variation of V_2 discounted by the risk-free interest rate is

$$d_t V_2(t) = \int_t^{\infty} du\ h(t,u)[d_t P(t,u) - r(t)P(t,u)dt]. \quad (52)$$

There are no other terms in (52) as a sale or purchase of bonds and deposit or withdrawal on the bank account correspond only to a change of the nature of the investment but not to a change of wealth. The maturity of a bond is not a change of wealth either. Only the variations of the bond prices have to be taken into account. Note that the term $d_t B(t) - r(t)B(t)dt$ vanishes identically from the definition of the spot rate and thus the amount invested in the bank account does not contribute to the discounted variation of V_2. The standard replication argument [23] for option pricing and hedging amounts to equate the variations in values of the two portfolios. We need some more ingredients before carrying out this program (see the appendix for a pedagogical exposition of this formulation for the standard European call option problem). See reference [24] for a different approach in the non-Gaussian case.

We assume that bond prices are driven by one of the stochastic string processes $Z(t,x)$ introduced in [12] and used above. $C(t)$ is *a priori* a function of a continuous infinity of bond prices at time t. The relevant mathematical tool to calculate $d_t C$ is that of functional derivation. We have, up to order dt,

$$\begin{aligned} d_t C \ =\ & \frac{\partial C}{\partial t}dt + \int_t^{\infty} du\ \frac{\partial C}{\partial P(t,u)} d_t P(t,u) \\ & + \frac{1}{2}\int_t^{\infty} du \int_t^{\infty} dv \frac{\partial^2 C}{\partial P(t,u)\partial P(t,v)} \\ & \times \mathrm{Cov}\left[d_t P(t,u),\ d_t P(t,v)\right] \\ & + \frac{\partial C}{\partial r} d_t r + \frac{1}{2}\frac{\partial^2 C}{\partial r^2}\mathrm{Var}[d_t r] \\ & + \frac{1}{2}\int_t^{\infty} du \frac{\partial^2 C}{\partial r \partial P(t,u)} \mathrm{Cov}\left[d_t P(t,u),\ d_t r(t)\right]. \end{aligned} \quad (53)$$

In (53), we have taken into account that the option price C is also a function of the stochastic spot rate $r(t)$. $d_t P(t,s)$ is given by [12]

$$\begin{aligned} d_t P(t,s) = dt\ P(t,s) \Bigg[& f(t,0) \\ & - P(t,s) \int_0^{s-t} dy\ \sigma(t,y)\ d_t Z(t,y) \Bigg]. \end{aligned} \quad (54)$$

This expression can be derived directly from (16), (27) and Ito's calculus. Thus

$$\begin{aligned} \mathrm{Cov}\ \ [d_t P(t,u), d_t P(t,v)] = & P(t,u)\ P(t,v) \\ & \int_0^{u-t} dy\ \sigma(t,y) \int_0^{v-t} dy'\ \sigma(t,y') \\ & \times \mathrm{Cov}\left[d_t Z(t,y),\ d_t Z(t,y')\right]. \end{aligned} \quad (55)$$

We also have

$$\mathrm{Var}[d_t r] = [\sigma(t,0)]^2 \mathrm{Var}[d_t Z(t,0)], \quad (56)$$

and

$$\begin{aligned} \mathrm{Cov}\ \ [d_t P(t,s), d_t r(t)] = & -\sigma(t,0)\ P(t,s) \int_0^{s-t} dy\ \sigma(t,y) \\ & \times \mathrm{Cov}\left[d_t Z(t,y),\ d_t Z(t,0)\right]. \end{aligned} \quad (57)$$

The portfolio of bonds replicates the option if $dV_1(t) = dV_2(t)$. We already see that a necessary condition for the replication to be perfect (the market to be complete) is that the replicating portfolio is made of a continuous infinity of bonds. Technically, this comes from the fact that the time differential of the option price C leads, by the functional differential, to a continuous integral over all bonds with time-to-maturity larger than t when the payoff is indeed dependent on the continuous infinity of bond maturities.

[13] Expressing the payoff as a function of the full forward curve is the same as expressing it as a function of the corresponding continuous set of bonds.

The stochastic part proportional to $d_tP(t,u)$ in the replicating equation $dV_1(t) = dV_2(t)$ cancels out if we choose

$$h(t,u) = \frac{\partial C(t)}{\partial P(t,u)}, \quad (58)$$

which is the usual delta hedging. But this is not enough as the stochastic term proportional to d_tr still remains. From the fact that $r(t) \equiv f(t,0)$ and by definition

$$P(t,s) = \exp\{-\int_0^{s-t} dy\ f(t,y)\}, \quad (59)$$

we see that bonds close to maturity are driven by the same stochastic innovations as the spot rate. To make sense of this statement, one has to be careful on how the limit $s \to t$ is taken. The standard way to tackle this problem is to remember that the continuous formulation is nothing but a limit $\delta t \to 0$ of discrete time increments. We thus have, instead of (59),

$$P(t,s) = \exp\{-\delta t\ [f(t,0) + f(t,\delta t) \\ +f(t,2\delta t) + ... + f(t,(n-1)\delta t)]\}, \quad (60)$$

where $n = (s-t)/\delta t$. In particular,

$$P(t,t+\delta t) = \exp\{-\delta t\ f(t,0)\} = 1 - \delta t\ f(t,0), \quad (61)$$

where the second equality becomes asymptotically exact for very small δt. This shows that it is possible in principle to hedge the spot rate by bonds that are very close to maturing. From (61), we obtain

$$d_tP(t,t+\delta t) = -\delta t\ d_tf(t,0) = -\delta t\ d_tr(t). \quad (62)$$

From this, we see that the stochastic part proportional to $d_tr(t)$ in the replicating equation $dV_1(t) = dV_2(t)$ cancels out if we add the quantity of bonds

$$\delta h(t,t+\delta t) = -\frac{1}{\delta t}\frac{\partial C(t)}{\partial r(t)}, \quad (63)$$

to the previous quantity $h(t,t+\delta t) = \frac{\partial C(t)}{\partial P(t,t+\delta t)}$ obtained from (58) for the bonds going to maturity. Notice that the two quantities are equal since one can replace $-\delta t \partial r(t)$ by $\partial P(t,t+\delta t)$ as seen from (62).

To summarize, the hedging strategy is given by

$$h(t,u) = [2 - Y(u-\delta t)]\ \frac{\partial C(t)}{\partial P(t,u)}, \quad (64)$$

where $Y(x)$ is the Heaviside function equal to 1 for $x \geq 1$ and zero otherwise. Thus, we recover the usual delta hedging, except for the factor 2 for bonds close to maturation which results from the existence of a stochastic spot interest rate.

The *deterministic* equation of the option price is then

$$\frac{1}{2}\int_t^\infty du \int_t^\infty dv\ A(t,u,v)\ \frac{\partial^2 C(t)}{\partial P(t,u)\partial P(t,v)}P(t,u)\ P(t,v) \\ + \frac{1}{2}\int_t^\infty du \frac{\partial^2 C}{\partial r \partial P(t,u)}\mathrm{Cov}\,[d_tP(t,u),\ d_tr(t)] \\ + \frac{\mathrm{Var}[d_tr]}{2}\frac{\partial^2 C}{\partial r^2} + \frac{\partial C(t)}{\partial t} - r(t)C(t) \\ + r(t)\int_t^\infty du\ [2 - Y(u-\delta t)]\ \frac{\partial C(t)}{\partial P(t,u)}P(t,u) = 0 \quad (65)$$

where

$$A(t,u,v) \equiv \int_0^{u-t} dy\ \sigma(t,y) \\ \int_0^{v-t} dy'\ \sigma(t,y')c_Z(t,y,y'). \quad (66)$$

This equation is correct only when $\sigma(t,x)$ is not an explicit function of the forward rates f, a case corresponding to linear SPDE's (37). When this is not the case, *i.e.* when $\sigma(t,x)$ is a function of f, we see from (65) with (66) that C should also be a function of f, in addition to be dependent on $t, \{P(s)\}$ and $r(t)$. As a consequence, the total time derivative d_tC must contains terms involving partial derivatives with respect to f that must be treated self-consistently with the other terms in (54).

8.2 Bond derivatives

The general case simplifies greatly when the contingent claim has a payoff that can be written as a function of a single bond price. We see that pricing and hedging these derivatives is much easier. This is very interesting since most derivatives that are actually traded have payoffs that can be written as a function of a finite number of bond prices.

The simplest case of a bond option is a European call or put on a zero-coupon bond. Consider a call with maturity s on a bond with maturity $s+\tau$, with strike price K. The payoff at maturity is the amount

$$C(s,P(s,s+\tau)) = \max(P(s,s+\tau) - K, 0), \quad (67)$$

The price of this claim at time $t < s$ will only be a function of $P(t,s+\tau)$, and the derivative can be hedged with a position in this bond alone.

In the same manner, caps[14], floors[15], collars[16] and swaptions[17] have payoffs that, in general, can be written

[14] A cap guarantees a maximum interest rate for borrowing over a determined time horizon.

[15] A floor guarantees a minimum interest rate for an investment over a determined time horizon.

[16] A collar is a contract in which the buyer is guaranteed an interval of interest rates, with a maximum rate. Its sale is thus the association of the buy of a cap and the sell of a floor.

[17] A swaption is an option on a swap. Simply put, an interest rate swap is a contract between two parties that exchange for a determined time two interest rates, for instance a short-term and a long-term.

as a function of the prices of a finite set of bonds. The simplest representative example of this type of options is a caplet. A caplet (settled in arrears) pays at maturity, the maximum of the LIBOR (London Interbank Offered Rate) for the caplet period (set at the beginning of the period) minus the cap rate and zero, multiplied by the principal amount. Let the current time be t, the beginning of the caplet period be s and the length of the caplet period be τ. Denote the principal amount by V, let the LIBOR be $L(s, s+\tau)$, and the cap rate be K. Then, the payoff at date $s+\tau$ will be

$$C(s+\tau) = V\tau \max(L-K, 0),$$

that we can express in our notation as

$$C(s+\tau) = V \max\left(e^{\int_0^\tau f(s,y)dy} - 1 - K\tau, 0\right)$$

$$= V \max\left(\frac{1}{P(s,s+\tau)} - 1 - K\tau, 0\right).$$

Finally, this payoff is known at time s, so we can write it as

$$C(s) = V \max\left(1 - (1+K\tau)P(s,s+\tau), 0\right)$$

Again, the payoff only depends on the price of a single bond, so that the claim can be priced and hedged with that bond.

If the time t value of the contingent claim depends only on t and $P(t,s)$, the replicating portfolio can have only the bond $P(t,s)$. Then, the previous calculation simplifies as all functional derivations transform into the usual derivation and we get, instead of (65):

$$\begin{aligned} &\frac{1}{2}\ A(t,s)[P(t,s)]^2 \frac{\partial^2 C}{\partial [P(t,s)]^2} \\ &+ \frac{1}{2}\frac{\partial^2 C}{\partial r(t) \partial P(t,s)} \mathrm{Cov}\left[d_t P(t,s),\ d_t r(t)\right] \\ &+ \frac{\mathrm{Var}[d_t r]}{2}\frac{\partial^2 C}{\partial r^2} \\ &+ r(t)P(t,s)\frac{\partial C}{\partial P(t,s)} \\ &+ r(t)P(t,t+\delta t)\frac{\partial C}{\partial P(t,t+\delta t)} \\ &+ \frac{\partial C}{\partial t} - r(t)C = 0 \end{aligned} \qquad (68)$$

where

$$A(t,s) = \int_0^{s-t} dy\ \sigma(t,y) \int_0^{s-t} dy'\ \sigma(t,y')\ c_Z(t,y,y'). \qquad (69)$$

The equation (68) is similar to the usual Black-Scholes equation but has two differences. First it has a time-dependent diffusion coefficient $A(t,x)$. But more importantly, it shows that the option price is function of *two* bond prices, the bond underlying the writing of the option and the bond just before maturation. We see that the expression (68) has *a priori* no mathematical sense in the continuous limit but takes full sense when we replace all derivatives by their discrete differences. This is a novel situation brought about by the structure of our model defined in terms of correlated but different shocks for each maturities as seen from equation (27). In fact, the continuous limit can be retrieved by remarking that $\frac{\partial C}{\partial P(t,t+\delta t)}$ should be of order δt:

$$\frac{\partial C}{\partial P(t,t+\delta t)} = \delta t\ w(t, P(t,s)). \qquad (70)$$

Then, the term $r(t)P(t,t+\delta t)\frac{\partial C}{\partial P(t,t+\delta t)}$ in (68) becomes $-r(t)\ w(t,P(t,s))\ P(t,t+\delta t)\ \frac{\partial C}{\partial r(t)}$. This situation is similar to a boundary layer for singular perturbation problems and in hydrodynamics where a special treatment has to be developed close to a boundary, here the spot maturity.

This equation contrasts with the pricing PDE that is usually presented in term structure models. In models with state variables such as CIR, the objective is to price derivatives as a function of those state variables and time, and so the PDE is set with respect to them. The solution of the PDE we present is not a function of state variables, which do not exist in our model, but rather is a function of the price of the bond underlying the derivative.

It is interesting to compare this pricing equation (68) with (69) to the corresponding pricing equation for the standard HJM model with a single Brownian motion driving the full forward rate curve. In this case, we have

$$\begin{aligned} d_t P(t,s) = P(t,s)\Bigg\{&\left[\phi(t)\int_0^{s-t}\sigma(t,y)\ dy\right]dt \\ &+ dW(t)\int_0^{s-t} dy\ \sigma(t,y)\Bigg\}, \end{aligned} \qquad (71)$$

where $\phi(t)$ is the market price of risk [15].

Thus

$$\mathrm{Var}\left[d_t P(t,s)\right] = dt\ [P(t,s)]^2 \left(\int_0^{s-t}\sigma(t,y)\ dy\right)^2. \qquad (72)$$

Using the same method as above, except for the term $r(t)P(t,t+\delta t)\frac{\partial C}{\partial P(t,t+\delta t)}$ which is absent, we get the same pricing equation (68) with the instantaneous "diffusion coefficient"

$$A(t,s) = \left(\int_0^{s-t}\sigma(t,y)\ dy\right)^2, \qquad (73)$$

instead of (69). The expression (69) reduces to (73) for $c_Z = 1$ as it should, *i.e.* for perfect correlations along the time-to-maturity axis, which corresponds to a single Brownian process (single factor) driving the whole forward rate curve.

9 Conclusion

All previous models of interest rates envision the fluctuations of the forward rate curve as driven by one or several (multi-factors) “zero-dimensional” random walk processes acting either on the left-end-point of the string, the short-rate, or on the whole function $f(t, x)$ simultaneously. To sum up the situation pictorially, in the first class of models, the forward rate curve is like a whip held by a shaking hand (the central banks!?), while in the second class, imagine a large hammer of the size of the full curve hitting it simultaneously through an irregular pillow (the volatility curve) transmitting inhomogeneously the impact along the curve.

The present paper, which extends reference [12], explores the more general situation where all the points of the forward curve are simultaneously driven by random shocks. Pictorially, this is like a string submitted to the incessant impacts of rain drops. We have argued that the condition of absence of arbitrage opportunity provides a constructive principle for the establishment of a general theory of interest rate dynamics. We have derived the general condition (25) that a stochastic partial differential equation (SPDE) for the forward rate must obey. Using our previous parameterization in terms of string shocks [12], we have derived the general structure (37) of the SPDE for forward rates under the no-arbitrage condition. It is noteworthy that the no-arbitrage condition excludes the parabolic class of partial differential equations, *i.e.* those that usually describe the physical strings submitted to strong fluctuations!

In a second part, we have derived a general approach to price and hedge options defined on bonds and thus on the forward rate curve, in terms of functional partial differential equations. We have found that the usual Black-Scholes replication strategy can be adapted to this situation, provided that special terms be added to account self-consistently for the instantaneous interest rate. Our approach is limited to the case where the volatility $\sigma(t, x)$ does not depend on $f(t, x)$ itself, thus excluding the a priori most interesting class of stochastic *non-linear* partial differential equations.

Let us end with a conjecture. It may be conceivable that a simple form for the nonlinear dependence of σ as a function of $f(t, x)$, with one or two real parameters, might embody the full phenomenology of observed forward rate statistics, hence constituting a truly fundamental theory of forward rate curves. It would be fundamental in the sense that the properties of the shocks would be generated dynamically by the nonlinear interactions similarly for instance to the coherent vortices in turbulence for instance, and in contrast to the usual external shock assignments in the existing linear models. The complex behavior would then result from the interplay between external factors represented by the noise source and the non-linear dynamics. Persuing along this conjectural tone, this approach might provide an line of attack for understanding the recent empirical finding [25] of a causal information cascade across scales in volatilities, occurring from large time scales to short times scales in a way very similar to the celebrated Kolmogorov energy cascade proposed for fluid turbulence.

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Appendix

The simplest option pricing problem (the so called ‘European call options’) is the following: suppose that an operator wants to buy from a bank a given share, a certain time $t = T$ from now ($t = 0$), at a fixed ‘striking’ price x_c. If the share value at $t = T$, $x(T)$, exceeds x_c, the operator ‘exercises’ his option. His gain, when reselling immediately at the current price $x(T)$, is thus the difference $x(T) - x_c$. On the contrary, if $x(T) < x_c$ the operator does not buy the share. What is the price $\mathcal{C}$ of this option, and what trading strategy ϕ should be followed by the bank between now and T, depending on what the share value $x(t)$ actually does between $t = 0$ and $t = T$?

In the standard treatment in continuous time [23], one forms the portfolio $F = -\mathcal{C} + x\phi$ such that d_tF remains indentically zero. Here, we propose a slightly different derivation of Black-Scholes’ result based on the idea that the bank constructs a portfolio that replicates the option exactly, thereby eliminating all risks. This is of course only valid under the restricted assumptions of continuous trading, Gaussian random walk of market prices, and absence of market imperfections and transaction costs.

The idea is to compare the two points of view of the option buyer and of the option seller. During the time increment dt, their respective change of wealth discounted by the risk-free interest rate r is

$$dW_a = d\mathcal{C} - r\mathcal{C}dt, \tag{74}$$

for the buyer who owns only the option and

$$dW_v = \phi[dx - rxdt], \tag{75}$$

for the seller who possesses a portfolio made of ϕ underlying stock shares. $d\mathcal{C} - r\mathcal{C}dt$ is the gain or loss of the buyer above the risk-free return. $\phi[dx - rxdt]$ is the gain or loss of the seller for a price variation of the stock above the risk-free return. The fair price of the option and the hedging strategy that the seller must follow are those such that the return is the same for both traders, namely

$$dW_a = dW_v. \tag{76}$$

This equality makes concrete the fundamental idea of Black and Scholes that the fair price and the hedging strategy are univoquely determined for all traders, independently of their risk aversion, if the seller can replicate the option.

Using Ito’s formula to calculate $d\mathcal{C}$, we have:

$$d\mathcal{C} = \frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial t} dt + \frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial x} dx + \frac{D}{2} \frac{\partial^2 \mathcal{C}(x, x_c, T-t)}{\partial x^2} dt. \tag{77}$$

Inserting this expression in (76) shows immediately that, if one makes the choice

$$\phi = \phi^* \equiv \frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial x}, \qquad (78)$$

then the only stochastic term, namely dx, cancels out! The comparison between the two returns of the buyer and seller become certain. In this case, the equation (76) provides the following deterministic partial differential equation for $\mathcal{C}$:

$$\frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial t} + rx \frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial x} + \frac{D}{2} \frac{\partial^2 \mathcal{C}(x, x_c, T-t)}{\partial x^2} - r\mathcal{C}(x, x_c, T-t) = 0, \qquad (79)$$

with boundary conditions the value of the option at maturity $C(x, x_c, 0) \equiv \max(x - x_c, 0)$. The solution of this equation is the celebrated Black and Scholes formula [10]. The hedging strategy is then the derivative of this solution with respect to x. This derivation shows very straightforwardly why the average return of the underlying stock does not appear; it has been avoided by the replication condition (76) with (78).

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